On the Origin of Kaluza-Klein Structure

J. Madore

*Laboratoire de Physique Théorique et Hautes Energies**
Université de Paris-Sud, Bât. 211, F-91405 ORSAY

J. Mourad

Laboratoire de Modèles de Physique Mathématique
Parc de Grandmont, Université de Tours, F-37200 TOURS

Abstract: It is suggested that quantum fluctuations of the light cone are at the origin of what appears at low energy to be a higher-dimensional structure over space-time. A model is presented which has but a finite number of Yang-Mills fields although the supplementary algebraic structure is of infinite dimension.

LPTHE Orsay 95/00

hep-th/9506041

March, 1995

* *Laboratoire associé au CNRS.*

1 Introduction

The fact that Yang-Mills potentials can be unified with the gravitational potential to build the components of a metric in a higher-dimensional manifold can be taken as evidence that in some sense the effective dimension of space-time is in fact larger than four. A modification of the traditional realization of this argument has been proposed in which the additional dimensions are replaced by an algebraic structure and an associated differential calculus which is not necessarily derived from the traditional calculus of a differential manifold. That is, it has been proposed that the extra structure should be described by a noncommutative geometry. (Dubois-Violette *et al.* 1989, Madore 1990, Chamseddine *et al.* 1993, Madore & Mourad 1993). This means that the Kaluza-Klein unification can be achieved using a space-time of dimension four. There remains however the problem of explaining the extra algebraic structure. We suggest here that its origin is to be found in the quantum fluctuations of the gravitational field and that it is essentially a quantum effect. Specifically, we show that in spite of the fact that the quantum fluctuations would be expected to give rise to an infinite-dimensional algebra and hence *a priori* to an infinite number of Yang-Mills potentials, a model can be proposed which has only a finite number.

We are not in a position to give a space-time description of the effect we wish to describe and so we shall restrict our considerations to an arbitrarily chosen causal space-like slice V. Since our interest is only in the local structure of space-time, it is not a restriction to suppose that such a V exists. Once we have argued that V acquires an extra noncommutative structure we must suppose that time evolution will induce a similar structure on all of space-time.

We shall see that the restriction on the dimension of the space of possible Yang-Mills potentials stems from the fact that our definition of a linear connection makes essential use of the bimodule structure of the space of 1-forms. This accounts also for the difference of some of our conclusions from those of authors (Chamseddine *et al.* 1993, Sitarz 1994, Klimčík *et al.* 1994, Landi *et al.* 1994) who define a linear connection using the classical (Koszul 1960) formula for a covariant derivative on an arbitrary left (or right) module. A more detailed comparison of the two approaches is given in Sitarz (1995). We refer to Bailin & Love (1987) for an introduction to standard Kaluza-Klein theory and, for example, to Madore & Mourad (1993) for a motivation of the generalization to noncommutative geometry. By a connection we shall always mean here a metric-compatible torsion-free linear connection.

In Section 2 we give a short review of the algebraic version of Kaluza-Klein theory and its possible relation to quantum fluctuations. The material here is mainly speculative. It is partly an adaptation of the standard folklore (Deser 1957, Isham *et al.* 1971) on the role of the gravitational field as a universal regulator. In Section 3 we recall the definition of an extension of a linear connection within the context of noncommutative geometry. This is in part a recapitulation of the work of Kehagias *et al.* (1995). In Section 4 we present two models with an infinite-dimensional algebra as extra structure but with only a finite number of extra Yang-Mills modes.

2 Quantum fluctuations

Let V be a causal space-like slice of space-time and let $\mathcal{C}(V)$ be the (commutative, associative) algebra of smooth functions on V. A smooth, classical, scalar field defines by restriction an element of $\mathcal{C}(V)$. A quantum scalar field $f(x)$ is an operator-valued distribution on space-time. Although by restriction it does not define a distribution on V, if one smears it over an arbitrarily small time-like interval it defines a smooth function on V with values in the unbounded operators $\mathcal{L}$ on some Hilbert space (Borchers 1964). Formally it defines by restriction an element of an algebra $\mathcal{A}_0$ of functions on V with values in $\mathcal{L}$. If x and y are two distinct points of V they have space-like separation with respect to the Minkowski metric and $f(x)$ and $f(y)$ commute:

$$f(x)f(y) = f(y)f(x). \tag{2.1}$$

We can formally identify $\mathcal{A}_0$ with the tensor product of $\mathcal{C}(V)$ and a commutative subalgebra $\mathcal{L}_0$ of $\mathcal{L}$:

$$\mathcal{A}_0 = \mathcal{C}(V) \otimes_{\mathbb{C}} \mathcal{L}_0.$$

Suppose now that V is considered as fixed but that the space-time metric is allowed to vary. If the metric is such that two points x and y of V have no longer a space-like separation then (2.1) will no longer necessarily be valid. The commutation relations of a quantum scalar field depend critically on the space-time metric in which the field is quantized. Suppose in particular that because of quantum effects the metric fluctuates around the Minkowski metric. Let $\langle f(x) \rangle_g$ be the mean value of $f(x)$ taken over these fluctuations. Then it is to be expected that

$$\langle f(x)f(y) \rangle_g \neq \langle f(x)f(y) \rangle_g \tag{2.2}$$

even though x and y have a space-like separation with respect to the original Minkowski metric. We introduce a new product $*$ on $\mathcal{L}$ defined by

$$f_1(x) * f_2(y) = \langle f_1(x)\, f_2(y) \rangle_g \tag{2.3}$$

and we suppose that it is regulated by the gravitational fluctuations in the sense that the limit

$$f_1 * f_2(x) = \lim_{y \to x} f_1(x) * f_2(y) \tag{2.4}$$

is well defined. In general it is to be expected that

$$f_1 * f_2 \neq f_2 * f_1. \tag{2.5}$$

We define the (noncommutative, associative) algebra $\mathcal{A}$ to be the algebra of functions $\mathcal{A}_0$ but with the product (2.4). We shall in the following drop the $*$ in the product and we shall consider $\mathcal{A}$ to be the structure algebra of a classical noncommutative geometry. Let $\tilde{Z}^0$ be the center of $\mathcal{A}$. We shall assume that in some quasi-classical approximation we can identify $\tilde{Z}^0$ with the original algebra of functions:

$$\tilde{Z}^0 = \mathcal{C}(V). \tag{2.6}$$

The only other general property one could reasonably expect to know of $\mathcal{A}$ is that the commutator of two of its elements vanishes with some fundamental length which tends to zero with $\hbar(\text{Planck mass})^{-1}$. We shall however not explicitly use this fact.

From (2.6) it follows that there exists a differential algebra (Connes 1986) $\tilde{\Omega}^* = \Omega^*(\mathcal{A})$ over $\mathcal{A}$ which admits an imbedding

$$\Omega^*(V) \xrightarrow{i} \tilde{\Omega}^* \tag{2.7}$$

where $(\Omega^*(V), d)$ is the standard differential calculus over space-time. When necessary we shall distinguish the differential on space-time with a subscript V. Such an imbedding has been proposed (Dubois-Violette *et al.* 1989, Madore 1990, Madore & Mourad 1993, Kehagias *et al.* 1995, Madore 1995) as an appropriate noncommutative generalization of Kaluza-Klein theory. From (2.7) it follows that the 1-forms $\tilde{\Omega}^1$ can be written as a direct sum

$$\tilde{\Omega}^1 = \tilde{\Omega}^1_h \oplus \tilde{\Omega}^1_c \tag{2.8}$$

where, in the traditional language of Kaluza-Klein theory, Ω^1_h is the horizontal component of the 1-forms. It can be defined as the $\tilde{\Omega}^0$-module generated by the image of $\Omega^1(V)$ in $\tilde{\Omega}^1$ under the imbedding; it is given by

$$\tilde{\Omega}^1_h = \Omega^1(V) \otimes_{\mathbb{C}} \tilde{\Omega}^0. \tag{2.9}$$

The $\tilde{\Omega}^1_c$ is a complement of $\tilde{\Omega}^1_h$ in $\tilde{\Omega}^1$. A vector-space complement always exists. If we suppose that the quantum fluctuation which give rise to the $*$-product are about Minkowski space then there is an action of the Poincaré algebra on $\tilde{\Omega}^*$ and we can write the algebra $\tilde{\Omega}^0$ as a product

$$\tilde{\Omega}^0 = \mathcal{C}(V) \otimes_{\mathbb{C}} \Omega^0 \tag{2.10}$$

where Ω^* is some differential algebra. In this case we can choose

$$\tilde{\Omega}^1_c = \tilde{\Omega}^0 \otimes_{\mathbb{C}} \Omega^1 \tag{2.11}$$

and $\tilde{\Omega}^1_c$ is an $\tilde{\Omega}^0$-module complement of $\tilde{\Omega}^1_h$. The horizontal component of the 1-forms can be expected to have a more general significance whereas the complement depends on the Ansatz (2.10). The differential algebra Ω^* replaces the hidden manifold of traditional Kaluza-Klein theory.

Our starting point in the next section will be the Formula (2.6) which can be considered to be to a certain extent independent of the speculations which preceded it. Let $\tilde{Z}^1$ be the vector space of elements of $\tilde{\Omega}^1_c$ which commute with $\tilde{\Omega}^0$. The most important assumption we make is that the dimension of $\tilde{Z}^1$ is finite. This assumption is completely unmotivated. It is introduced to account for the fact that there are a finite number of observed Yang-Mills potentials. It replaces the truncation assumptions which are introduced in traditional Kaluza-Klein theories but it is a weaker assumption since in general the space $\tilde{\Omega}^1_c$ itself will be of infinite dimension as a vector space and in general not even finitely generated as a $\tilde{\Omega}^0$-module.

3 Covariant derivatives

Consider the case of a general (algebraic) tensor product

$$\tilde{\Omega}^* = \Omega'^* \otimes_{\mathbb{C}} \Omega''^* \tag{3.1}$$

of two arbitrary differential calculi Ω'^* and Ω''^* with corresponding covariant derivatives D' and D''. We have then two linear maps

$$\Omega'^1 \xrightarrow{D'} \Omega'^1 \otimes_{\Omega'^0} \Omega'^1, \qquad \Omega''^1 \xrightarrow{D''} \Omega''^1 \otimes_{\Omega''^0} \Omega''^1, \tag{3.2}$$

which satisfy the corresponding Leibniz rules. It has been stressed (Dubois-Violette & Michor 1995, Mourad 1995) that the appropriate generalization to noncommutative geometry of the Leibniz rule involves a generalized symmetry operation σ. We have then two maps

$$\Omega'^1 \otimes_{\Omega'^0} \Omega'^1 \xrightarrow{\sigma'} \Omega'^1 \otimes_{\Omega''^0} \Omega'^1, \qquad \Omega''^1 \otimes_{\Omega''^0} \Omega''^1 \xrightarrow{\sigma''} \Omega''^1 \otimes_{\Omega''^0} \Omega''^1$$

which are necessarily bilinear. In simple models, the quantum plane (Dubois-Violette *et al.* 1995) and some matrix models (Madore *et al.* 1995), the covariant derivative and the associated map σ have been shown to be essentially unique.

From D' and D'' we wish to construct an extension

$$\tilde{\Omega}^1 \xrightarrow{\tilde{D}} \tilde{\Omega}^1 \otimes_{\tilde{\Omega}^0} \tilde{\Omega}^1, \tag{3.3}$$

with its associated generalized symmetry operation $\tilde{\sigma}$. This is the most general possible formulation of the bosonic part of the Kaluza-Klein construction under the Ansatz (3.1). It has been shown (Kehagias *et al.* 1995) that the existence of a non-trivial extension places severe restrictions on the differential calculi of the two factors. These restrictions are at the origin of the fact that we can construct a Kaluza-Klein theory using an infinite algebra but still with only a finite number of Yang-Mills modes.

The 1-forms $\tilde{\Omega}^1$ can be written as in (2.8) with, in the present notation, (2.9) and (2.11) given by

$$\tilde{\Omega}^1_h = \Omega'^1 \otimes_{\mathbb{C}} \Omega''^0, \qquad \tilde{\Omega}^1_c = \Omega'^0 \otimes_{\mathbb{C}} \Omega''^1.$$

Let $f' \in \Omega'^0$, $f'' \in \Omega''^0$ and $\xi' \in \Omega'^1$, $\xi'' \in \Omega''^1$. Then it follows from the definition of the product in the tensor product that

$$f'\xi'' = \xi''f', \qquad f''\xi' = \xi'f''. \tag{3.4}$$

Hence one concludes that the extension $\tilde{\sigma}$ of σ' and σ'' which is part of the definition of $\tilde{D}$ is given by

$$\tilde{\sigma}(\xi' \otimes \eta'') = \eta'' \otimes \xi', \qquad \tilde{\sigma}(\xi'' \otimes \eta') = \eta' \otimes \xi''. \tag{3.5}$$

From these one deduces the constraints (Kehagias *et al.* 1995)

$$f'\tilde{D}\xi'' = (\tilde{D}\xi'')f', \qquad f''\tilde{D}\xi' = (\tilde{D}\xi')f'' \tag{3.6}$$

on $\tilde{D}$. These are trivially satisfied if

$$\tilde{D}\xi' = D'\xi', \qquad \tilde{D}\xi'' = D''\xi''. \tag{3.7}$$

Using the decomposition (2.8) one sees that the covariant derivative (3.3) takes its values in the sum of 4 spaces, which can be written in the form

$$\begin{aligned}
\tilde{\Omega}^1_h \otimes_{\tilde{\Omega}^0} \tilde{\Omega}^1_h &= (\Omega'^1 \otimes_{\Omega'^0} \Omega'^1) \otimes_{\mathbb{C}} \Omega''^0, \\
\tilde{\Omega}^1_h \otimes_{\tilde{\Omega}^0} \tilde{\Omega}^1_v &= \Omega'^1 \otimes_{\mathbb{C}} \Omega''^1, \\
\tilde{\Omega}^1_v \otimes_{\tilde{\Omega}^0} \tilde{\Omega}^1_h &= \Omega''^1 \otimes_{\mathbb{C}} \Omega'^1, \\
\tilde{\Omega}^1_v \otimes_{\tilde{\Omega}^0} \tilde{\Omega}^1_v &= \Omega'^0 \otimes_{\mathbb{C}} (\Omega''^1 \otimes_{\Omega''^0} \Omega''^1).
\end{aligned} \tag{3.8}$$

Let Z'^0 (Z''^0) be the center of Ω'^0 (Ω''^0) and let Z'^1 (Z''^1) be the vector space of elements of Ω'^1 (Ω''^1) which commute with Ω'^0 (Ω''^0). Then Z'^1 (Z''^1) is a bimodule over Z'^0 (Z''^0). Let Z'^2 (Z''^2) be the elements of $\Omega'^1 \otimes_{\Omega'^0} \Omega'^1$ $(\Omega''^1 \otimes_{\Omega''^0} \Omega''^1)$ which commute with Ω'^0 (Ω''^0). Then one finds the inclusion relations

$$Z'^1 \otimes_{Z'^0} Z'^1 \subset Z'^2, \qquad Z''^1 \otimes_{Z''^0} Z''^1 \subset Z''^2,$$

but in general the two sides are not equal. From (3.6) we see then that

$$\begin{aligned}
\tilde{D}\xi' &\in (\Omega'^1 \otimes_{\Omega'^0} \Omega'^1) \otimes_{\mathbb{C}} Z''^0 \oplus \Omega'^1 \otimes_{\mathbb{C}} Z''^1 \oplus Z''^1 \otimes_{\mathbb{C}} \Omega'^1 \oplus \Omega'^0 \otimes_{\mathbb{C}} Z''^2, \\
\tilde{D}\xi'' &\in Z'^0 \otimes_{\mathbb{C}} (\Omega''^1 \otimes_{\Omega''^0} \Omega''^1) \oplus \Omega''^1 \otimes_{\mathbb{C}} Z'^1 \oplus Z'^1 \otimes_{\mathbb{C}} \Omega''^1 \oplus Z'^2 \otimes_{\mathbb{C}} \Omega''^0.
\end{aligned} \tag{3.9}$$

In the relevant special case with $\Omega'^* = \Omega^*(V)$, we shall have

$$Z'^0 = \Omega'^0, \qquad Z'^1 = \Omega'^1, \tag{3.10}$$

and so (3.9) places no restriction on $\tilde{D}\xi''$.

The main assumption which we shall make is that

$$\dim_{\mathbb{C}}(Z''^1) < \infty. \tag{3.11}$$

We shall also suppose that

$$Z''^1 \otimes_{Z''^0} Z''^1 = Z''^2 \tag{3.12}$$

although it is easy to find pertinent cases where this would not be so. For example, the matrix geometry introduced by Connes & Lott (1992) has a vanishing left-hand side and a right-hand side of dimension 1. We refer to Kehagias *et al.* (1995) for details. Let θ^a be a basis of Z''^1 over the complex numbers. From (3.12) we can conclude that

$$d\theta^a = -\frac{1}{2} C^a{}_{bc} \theta^b \theta^c. \tag{3.13}$$

The product on the right-hand side is the product in the algebra Ω''^*. The $C^a{}_{bc}$ are elements of the algebra Ω''^0. Since the left-hand side commutes with all elements of the algebra they lie in the center Z''^0 of Ω''^0. We shall suppose that they are complex numbers.

We shall impose the condition that the connections be metric and without torsion although these might be considered rather artificial conditions on the components of the 1-forms in $\tilde{\Omega}^1_c$. We have then two bilinear maps

$$\Omega'^1 \otimes_{\Omega'^0} \Omega'^1 \xrightarrow{g'} \Omega'^0, \qquad \Omega''^1 \otimes_{\Omega''^0} \Omega''^1 \xrightarrow{g''} \Omega''^0, \tag{3.14}$$

which satisfy the compatibility condition (1.9), from which we must construct an extension

$$\tilde{\Omega}^1 \otimes_{\tilde{\Omega}^0} \tilde{\Omega}^1 \xrightarrow{\tilde{g}} \tilde{\Omega}^0$$

which satisfies also (1.9). From the decomposition (3.8) one sees that $\tilde{g}$ will be determined by two bilinear maps

$$\Omega'^1 \otimes_{\mathbb{C}} \Omega''^1 \xrightarrow{g_1} \tilde{\Omega}^0, \qquad \Omega''^1 \otimes_{\mathbb{C}} \Omega'^1 \xrightarrow{g_2} \tilde{\Omega}^0. \tag{3.15}$$

If $\tilde{g}$ is symmetric then from (3.5) it follows that

$$g_2 = g_1 \tilde{\sigma}.$$

In general it is to be expected that if the connection is metric and without torsion, the conditions (3.11) will place constraints also on the covariant derivative $\tilde{D}\xi''$.

In the relevant special case with $\Omega'^* = \Omega^*(V)$ one can define a metric $i^*\tilde{g}$ on V by

$$i^*\tilde{g}(\theta^\alpha, \theta^\beta) = \tilde{g}(\tilde{\theta}^\alpha, \tilde{\theta}^\beta), \quad \tilde{\theta}^\alpha = i(\theta^\alpha).$$

To maintain contact with the commutative construction of the previous section we suppose in this case that

$$i^*\tilde{g} = g_V. \tag{3.16}$$

Let θ^α be a (local) moving frame on V and set $\theta^i = (\theta^\alpha, \theta^a)$. The Kaluza-Klein construction follows as in Section 3 of Kehagias *et al.* (1995).

4 Models

There remains the task of constructing a reasonable model of a differential calculus which satisfies the conditions (3.11) and (3.12). Consider the case where the classical theory is invariant under an internal symmetry group G. Then the quantum theory can be defined to be also invariant under G. In particular, there is an action of G on the algebra $\mathcal{A}$. In order to be able to make use of previous calculations we shall consider only the case $G = SU_n$. Let λ_a be a basis of the Lie algebra of G and e_a the associated derivations of $\mathcal{A}$. There exists then (Dubois-Violette 1988) a differential calculus $(\tilde{\Omega}^*_D, \tilde{d}_D)$ based on the derivations such that $\tilde{\Omega}^1_D$ has a basis θ^a (Dubois-Violette *et al.* 1989, 1990) with the property that it commutes with the elements of $\mathcal{A}$. That is $\tilde{\Omega}^*_D$ is of the form (3.1) with

$$Z''^1_D = \{\theta^a\}. \tag{4.1}$$

Therefore (3.11) and (3.12) are satisfied. Let now $(\tilde{\Omega}^*, \tilde{d})$ be an arbitrary differential calculus which is an extension of $(\tilde{\Omega}^*_D, \tilde{d}_D)$ and which is such that

$$Z''^1 = Z''^1_D, \qquad Z''^2 = Z''^2_D. \tag{4.2}$$

Then (3.11) and (3.12) remain satisfied for $(\tilde{\Omega}^*, \tilde{d})$. The extended differential calculus would have to be 'large enough', be such that the condition $\tilde{d}f = 0$, for $f \in \mathcal{A}$ implies that $f = 1$. This would not be the case for $(\tilde{\Omega}^*_D, \tilde{d}_D)$.

As a second example let us consider the equilibrium physics of a particle, confined to a two-dimensional plane, which obeys anyon statistics. The space-time is of euclidean signature then and the quantum fluctuations of the light-cone are replaced by the equally complicated effects of solid-state physics. The classical wave function can be considered as a function on $\mathbb{R}^2$ with values in the quantum plane. We shall formally identify the algebra $\mathcal{A}$ of such functions with the tensor product of the algebra of functions on $\mathbb{R}^2$ and the algebra of the quantum plane. As differential calculus $\Omega^*(\mathcal{A})$ we can choose the tensor product of the form (3.1) with $\Omega'^* = \Omega^*(\mathbb{R}^2)$ and Ω''^* equal to the differential calculus on the quantum plane introduced by Pusz & Woronowicz (1989) and Wess & Zumino (1990). We have then a Kaluza-Klein theory over $\mathbb{R}^2$. Following the reasoning of Kehagias *et al.* (1995) one can show that there is only a trival Kaluza-Klein extension. This is not the case if one replaces the quantum plane with the noncommutative torus (Connes & Rieffel 1987, Connes 1994). The algebra $\mathcal{A}_\alpha$ of the latter is an infinite involutive unital algebra generated by two elements u and v subject to the relations:

$$vu = e^{2\pi i\alpha}uv, \qquad u^* = u^{-1}, \quad v^* = v^{-1}, \tag{4.3}$$

where α is an irrational number. The differential calculus based on the Dirac operator (Connes & Rieffel 1987) is equivalent to the differential calculus based on the outer derivations. These are the derivations modulo the inner derivations. They have two generators

$$e_1(u^n v^m) = inu^n v^m, \qquad e_2(u^n v^m) = imu^n v^m. \tag{4.4}$$

The space Ω''^1 is a free $\mathcal{A}_\alpha$-bimodule with a basis θ^a dual to e_a. An arbitrary 1-form may be written as $\omega = \omega_a\theta^a$ with ω_a two elements of $\mathcal{A}_\alpha$. It is easily verified that

$$Z''^1 = \{\theta^a\}, \qquad Z''^2 = \{\theta^a \otimes \theta^b\}, \tag{4.5}$$

and that

$$d\theta^a = 0. \tag{4.6}$$

The modified second example is similar to the first one but with the group G equal to $U_1 \times U_1$.

5 Conclusions

We have presented a rather general formulation of Kaluza-Klein theory in which the internal manifold is replaced by an abstract differential calculus with the unique restriction that the dimension of the space of 1-forms which commute with all elements of the algebra be finite. This restriction replaces the truncation assumption in traditional Kaluza-Klein theory. We have shown that with this assumption the number of Yang-Mills potentials is also finite in spite of the fact that the internal-structure algebra is otherwise quite arbitrary.

Acknowledgment: The first author would like to thank H. Goenner and the Heraeus Foundation for financial support during a three-month visit to the Institut für Theoretische Physik, Universität Göttingen where part of this research was done. He would like to thank A. Dimakis and F. Müller-Hoissen for some very lively (lebhafte) discussions there. He would like to thank also M. Dubois-Violette for enlightening discussions on algebraic field theory.

References

Bailin D., Love A. 1987, *Kaluza-Klein theories*, Rep. Prog. Phys. **50** 1087.

Borchers H.J. 1964, *Field Operators as $\mathcal{C}^\infty$ Functions in Spacelike Directions*, Il Nuovo Cimento **33** 1600.

Chamseddine A.H., Felder G., Fröhlich J. 1993, *Gravity in Non-Commutative Geometry*, Commun. Math. Phys. **155** 205.

Connes A. 1986, *Non-Commutative Differential Geometry*, Publications of the Inst. des Hautes Etudes Scientifique. **62** 257.

— 1994, *Non-commutative geometry*, Academic Press.

Connes A., Lott J. 1990, *Particle Models and Noncommutative Geometry*, in 'Recent Advances in Field Theory', Nucl. Phys. Proc. Suppl. **B18** 29.

— 1992, *The metric aspect of non-commutative geometry*, Proceedings of the 1991 Cargèse Summer School, Plenum Press.

Connes A., Rieffel M. 1987 *Yang-Mills for non-commutative two tori* Contemp. Math. **62** 237.

Deser S. 1957, *General Relativity and the Divergence Problem in Quantum Field Theory*, Rev. Mod. Phys. **29** 417.

Dubois-Violette M. 1988, *Dérivations et calcul différentiel non-commutatif*, C. R. Acad. Sci. Paris **307** Série I 403.

Dubois-Violette M., Michor P. 1994, *Dérivations et calcul différentiel non-commutatif II*, C. R. Acad. Sci. Paris **319** Série I 927.

— 1995, *Connections on Central Bimodules*, Preprint LPTHE Orsay 94/100.

Dubois-Violette M., Kerner R., Madore J. 1989 *Classical bosons in a noncommutative geometry*, Class. Quant. Grav. **6** 1709.

— 1990, *Non-Commutative Differential Geometry of Matrix Algebras*, J. Math. Phys. **31** 316.

Dubois-Violette M., Madore J., Masson T., Mourad J. 1994, *Linear Connections on the Quantum Plane*, Lett. Math. Phys. **0** 00.

Isham C.J., Salam A., Strathdee J. 1971, *Infinity Suppression in Gravity-Modified Quantum Electrodynamics*, Phys. Rev. **D3** 1805.

Kehagias A., Madore J., Mourad J., Zoupanos G. 1995, *Linear Connections on Extended Space-Time*, Tours Preprint, LMPM 95-01.

Klimčík C., Pompoš A., Souček. V. 1994, *Grading of Spinor Bundles and Gravitating Matter in Noncommutative Geometry*, Lett. Math. Phys. **30**, 259.

Koszul J.L. 1960, *Lectures on Fibre Bundles and Differential Geometry*, Tata Institute of Fundamental Research, Bombay.

Landi G., Nguyen Ai Viet, Wali K.C. 1994, *Gravity and electromagnetism in noncommutative geometry*, Phys. Lett. **B326** 45.

Madore J. 1990, *Modification of Kaluza-Klein Theory*, Phys. Rev. **D41** 3709.

— 1995, *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, Cambridge University Press.

Madore J., Masson T., Mourad J. 1994, *Linear Connections on Matrix Geometries*, Class. Quant. Grav. (to appear).

Madore J., Mourad J. 1993, *Algebraic-Kaluza-Klein Cosmology*, Class. Quant. Grav. **10** 2157.

Mourad. J. 1995, *Linear Connections in Non-Commutative Geometry*, Class. Quant. Grav. **12** 965.

Pusz W., Woronowicz S.L. 1989, *Twisted Second Quantization*, Rep. on Math. Phys. **27** 231.

Sitarz A. 1994, *Gravity from Noncommutative Geometry*, Class. Quant. Grav. **11** 2127.

— 1995, *On some aspects of linear connections in noncommutative geometry*, Jagiellonian University Preprint.

Wess J., Zumino B. 1990, *Covariant Differential Calculus on the Quantum Hyperplane* Nucl. Phys. B (Proc. Suppl.) **18B** 302.